Gauge Equivariant Mesh CNNs: Anisotropic Convolutions On Geometric Graphs

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Motivation and Main Problem

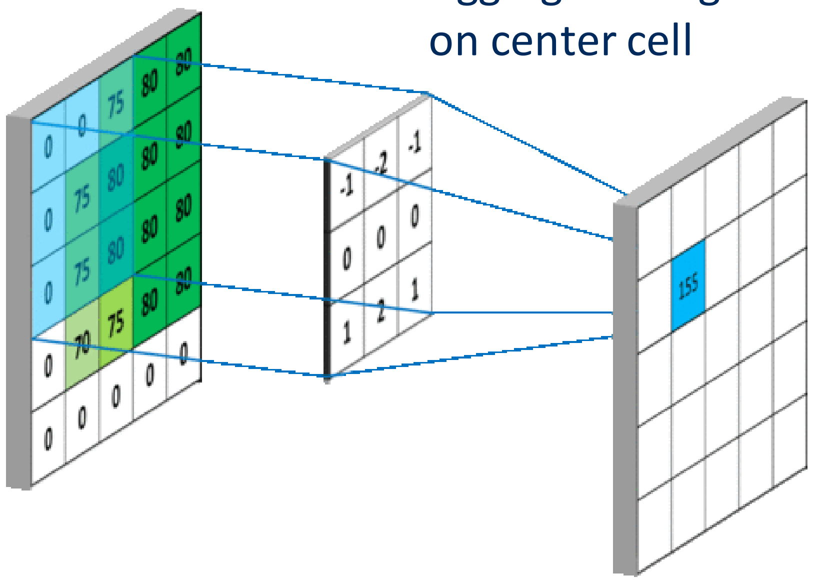
Convolutions are great in 2D for pattern recognition!

Convolutional Kernel

c_{11}	c_{12}	c_{13}
c_{21}	c_{22}	c_{23}
c_{31}	c_{32}	c_{33}

Slide over image

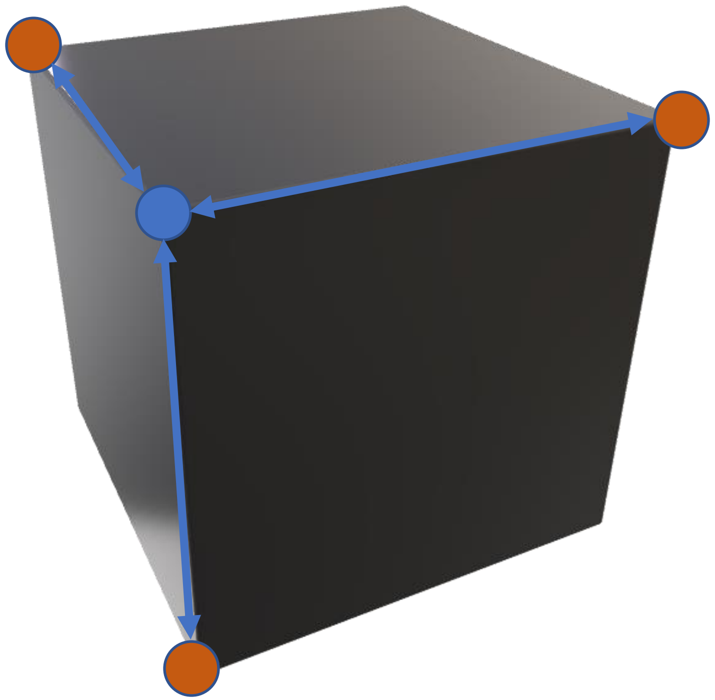
Aggregate neighboring features on center cell

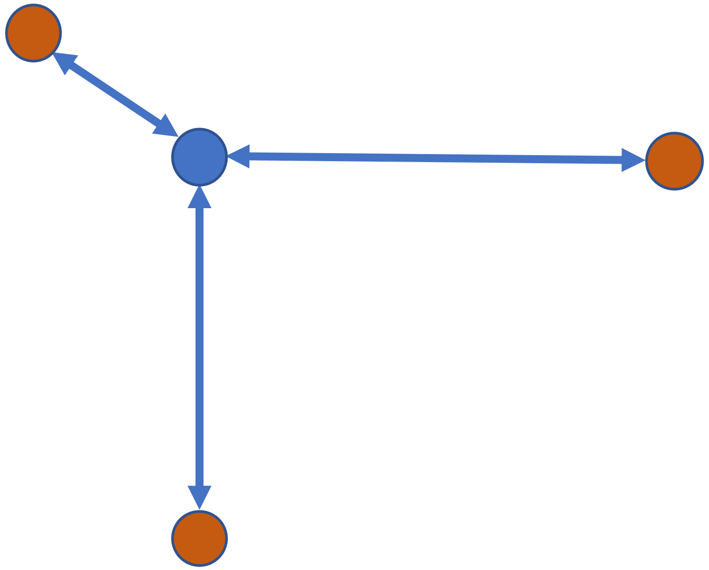


https://mlnotebook.github.io/post/CNN1/

Motivation and Main Problem

Convolutions on Meshes:

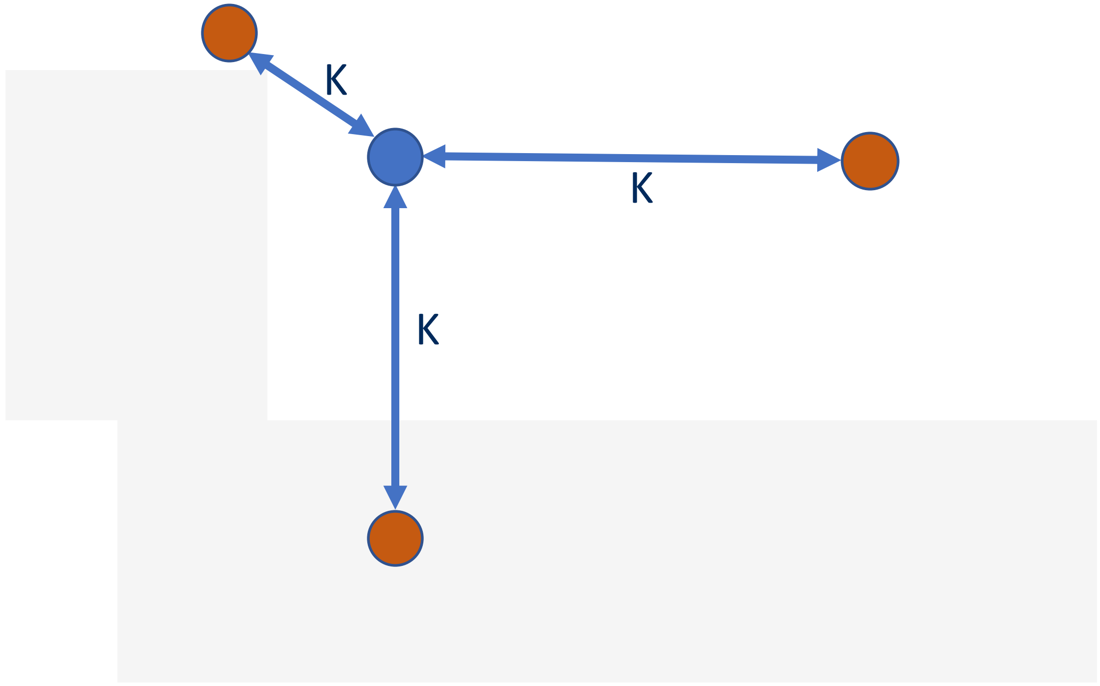




Motivation and Main Problem

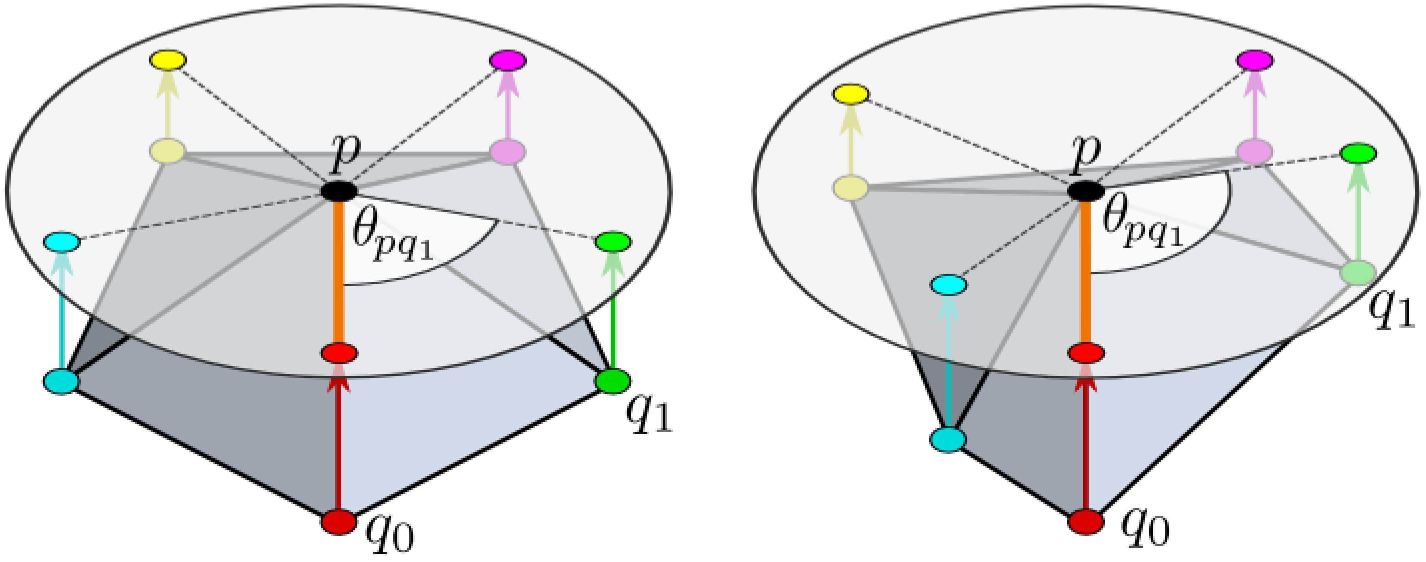
Scale features on neighbor vertices by the **same** kernel as each other

$$(K \star f)_p = \sum_{q \in \mathcal{N}_p} K_{\text{neigh}} f_q$$



Motivation and Main Problem

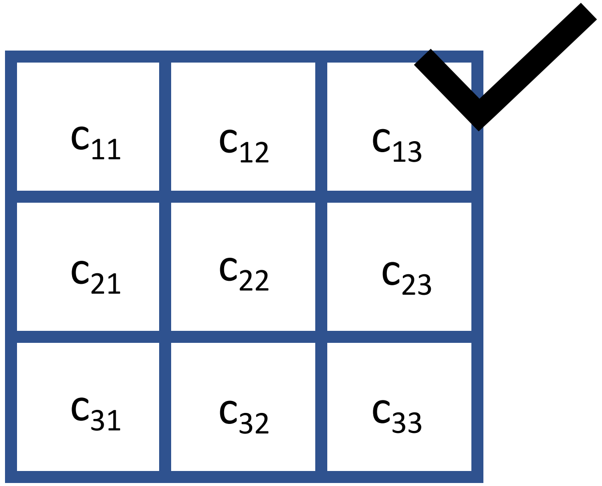
Convolutions on Meshes are not very expressive
because they are **isotropic**

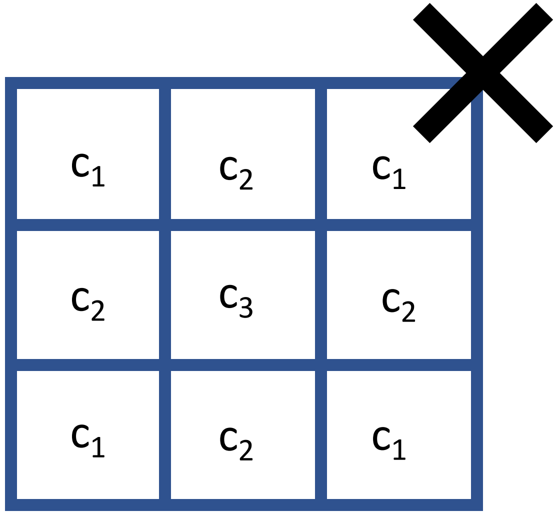


Motivation and Main Problem

If we designed **anisotropic** graph convolutional kernels, we could learn features more **efficiently**.

2D CNNs almost exclusively use **anisotropic** kernels.

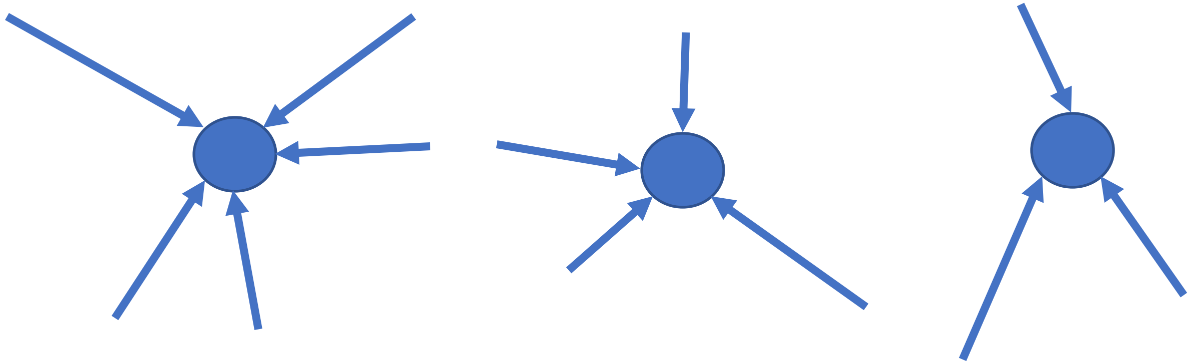




Motivation and Main Problem

Anisotropic kernels on meshes are difficult because:

- Arbitrary number edges incident on each vertex
- Edges are not ordered in any particular set order
- These edges can come from any arbitrary direction



c_{11}	c_{12}	c_{13}
c_{21}	c_{22}	c_{23}
c_{31}	c_{32}	c_{33}

Contributions

- Make convolutional kernels more expressive via anisotropy.
  - Allows for better learning of geometric features.
  - Difficulty in irregular/sparse structure of 3D geometry.
  - Prior work on anisotropic convs. only operate on flat/regular domains.
- Can embed anisotropy in a convolutional kernel by building a local reference frame on each vertex.
- Shows that anisotropy can more efficiently achieve SOTA performance on Shape Correspondence than other techniques.

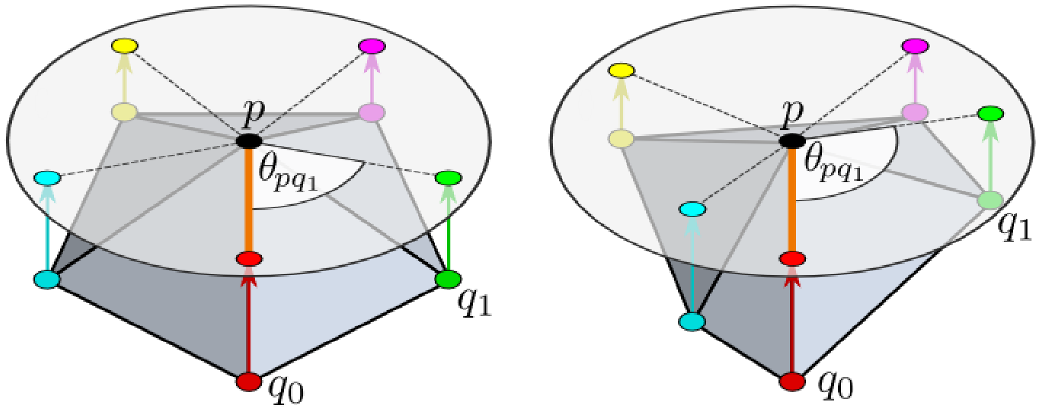
Problem Setting

We are looking for a convolution that looks like this, but that can distinguish the two neighborhoods.

N x 1 M x 1

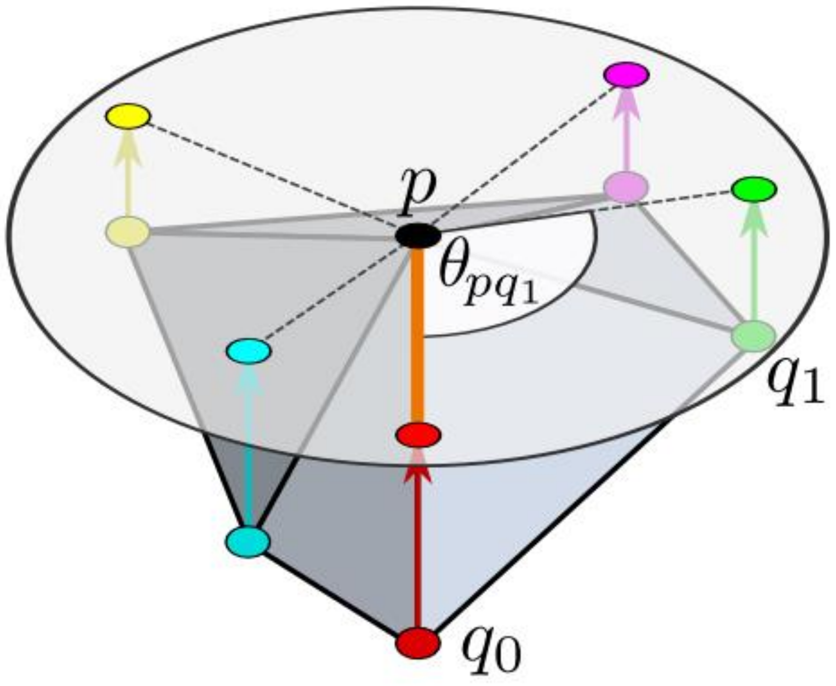
$$(K \star f)_p = \sum_{q \in \mathcal{N}_p} K_{\text{neigh}} f_q$$

N x M



Approach

Set up a **gauge** at each vertex by picking a **reference** edge



Each edge's kernel value now depends on angle w.r.t **gauge**.

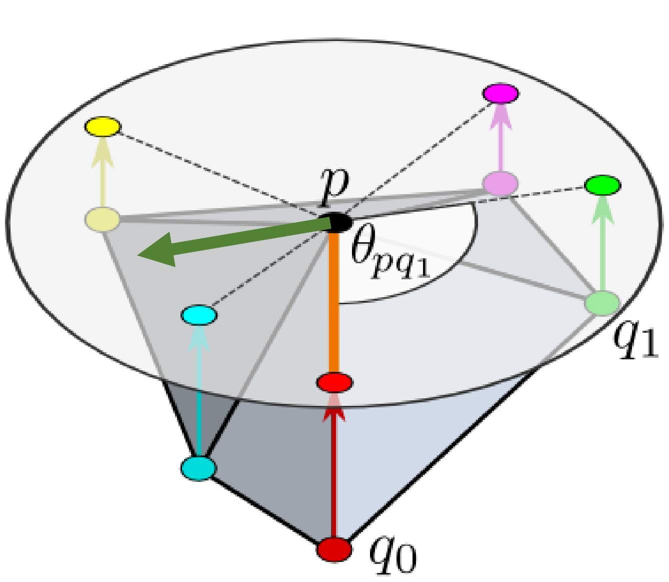
Angle measured on projection in **tangent** space

$$(K \star f)_p = \sum_{q \in \mathcal{N}_p} K_{\text{neigh}}(\theta_{pq}) \rho(g_{q \to p}) f_q$$

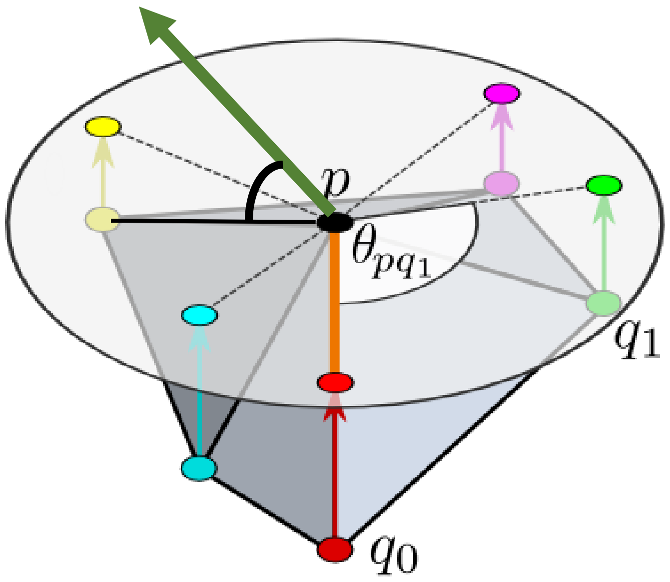
Features Living in Local Frame

Useful to think of m-d features living in a local frame defined by the **gauge** at each vertex.

2D feature vector

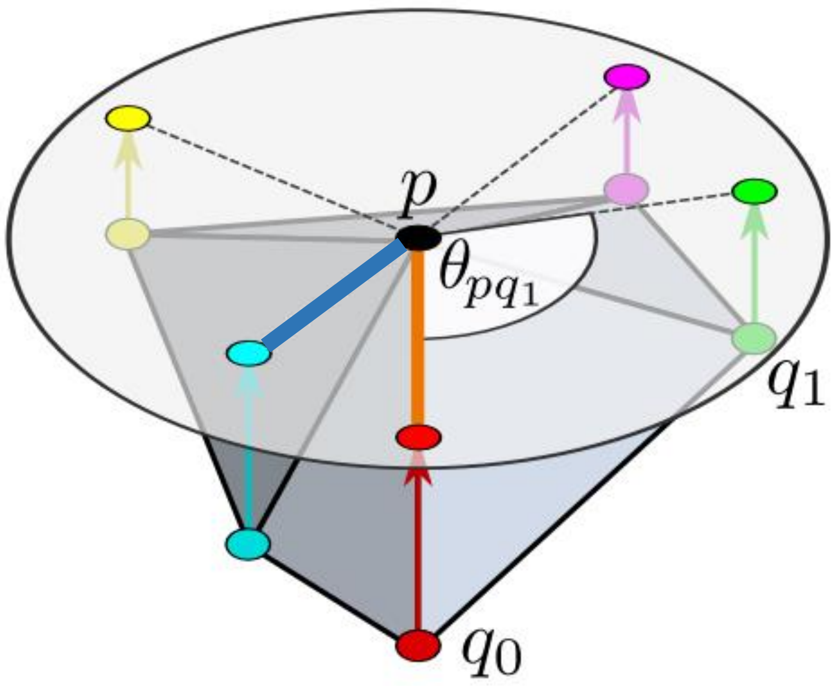


3D feature vector



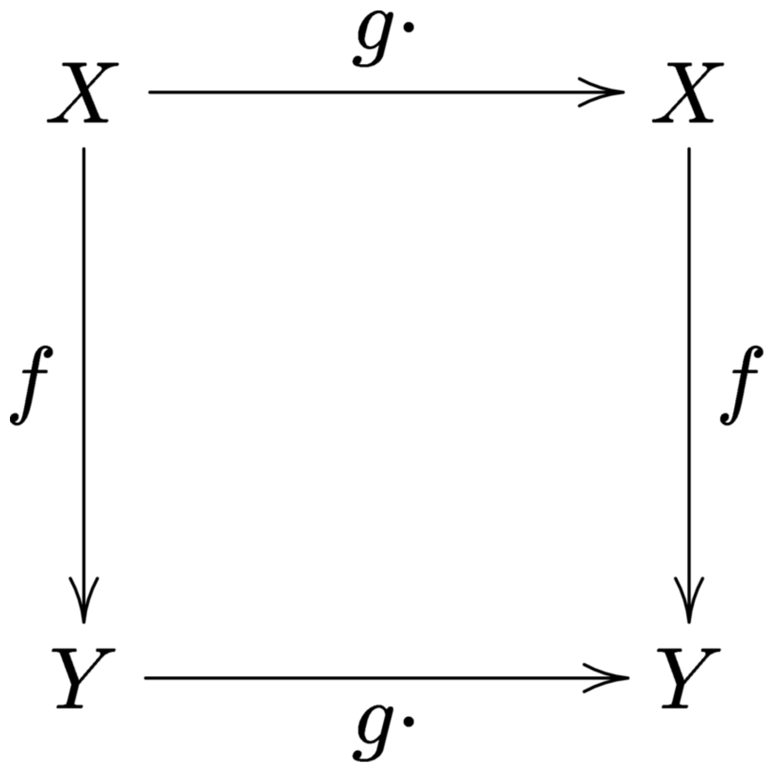
Big Problem

If you change the gauge, you change the output!



Equivariance

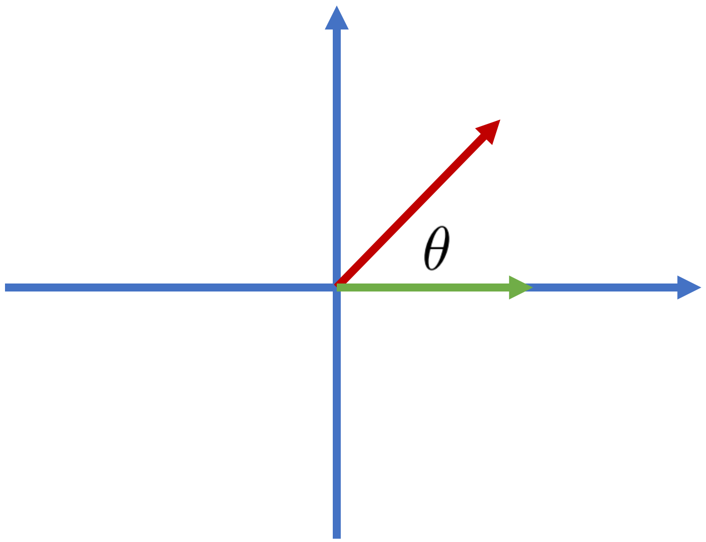
$$f(g\cdot x) = g\cdot f(x)$$



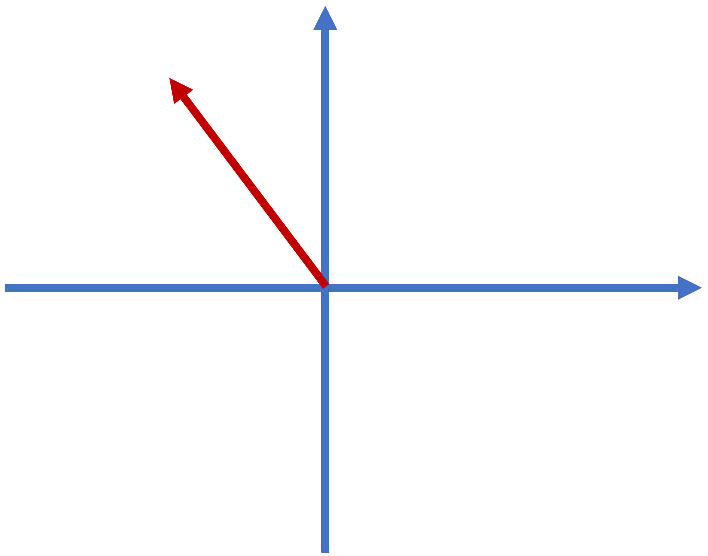
Gauge Equivariance 2D Output/1D Input

What is learned shouldn't be affected with choice of reference edge.

$$K_{\text{neigh}}(\theta - g) = R(-g) K_{\text{neigh}}(\theta)$$



Mesh Angle



Output feature

Gauge Equivariance 2D Output/1D Input

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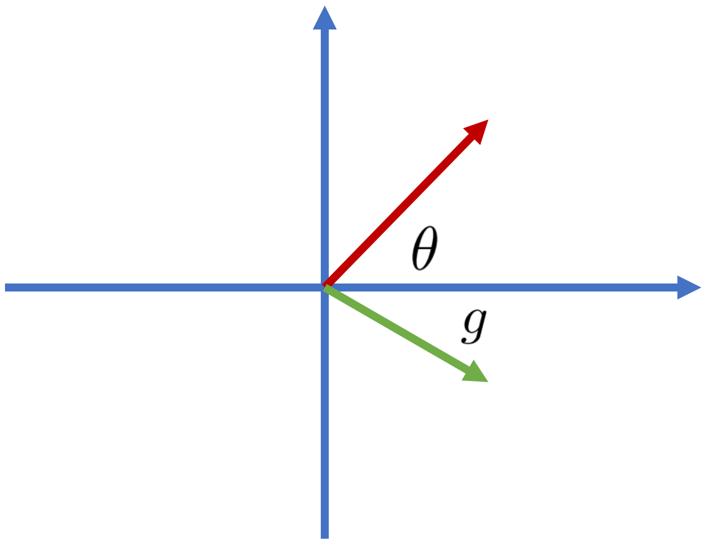
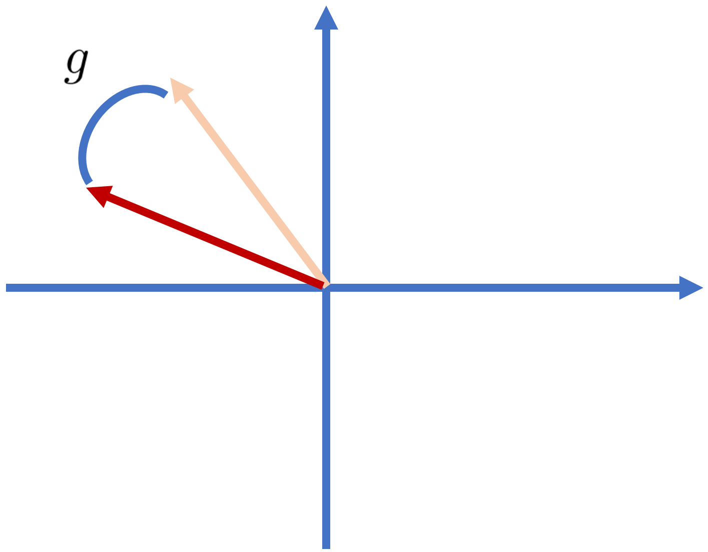


Mesh Angle



Output feature

General Gauge Equivariance

When given m-d input, with n-d output, must solve:

$$K_{\text{neigh}}(\theta - g) = \rho_{\text{out}}(-g) K_{\text{neigh}}(\theta) \rho_{\text{in}}(g)$$

General Gauge Equivariance

When given m-d input, with n-d output, must solve:

$$K_{\text{neigh}}(\theta - g) = \rho_{\text{out}}(-g) K_{\text{neigh}}(\theta) \rho_{\text{in}}(g)$$

How do we solve this?!

Group Representations of SO(2)

ρ_{out}, ρ_{in} Are representations of the SO(2) group of **planar rotations**

They take as input an angle, output an n-D planar rotation matrix :

e.g.

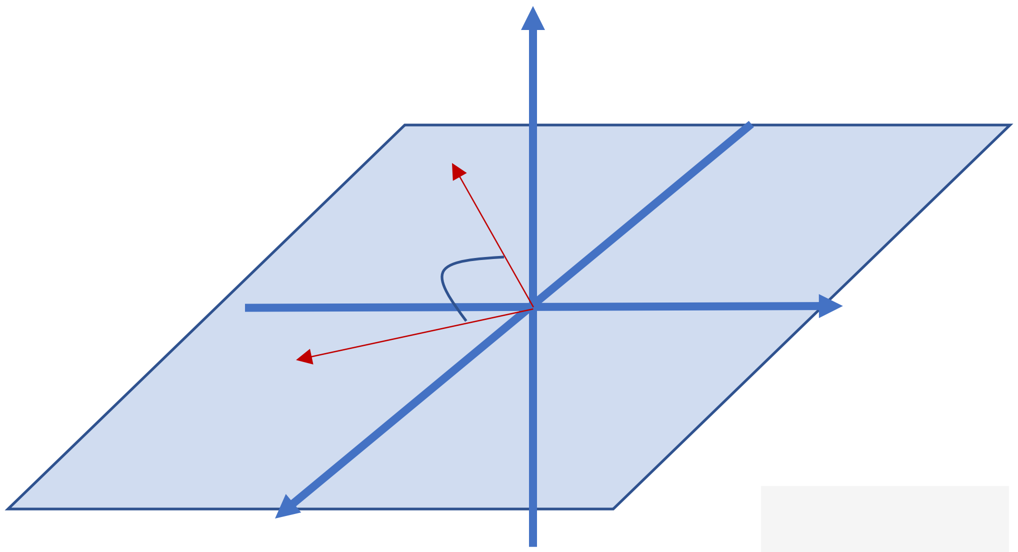
$$\rho(g) = \begin{pmatrix} 1 & & & & \\ & \cos g & -\sin g & & \\ & \sin g & \cos g & & \\ & & & \cos g & -\sin g \\ & & & \sin g & \cos g \end{pmatrix}$$

Group Representations of SO(2)

N x N SO(2) matrices rotate a vector in a plane.

Have a block diagonal structure.

$$\rho(g) = \begin{pmatrix} 1 & & \\ & \cos g & -\sin g \\ & \sin g & \cos g \end{pmatrix}$$



Rotations on 2D plane defined by **gauge**

Group Representations of SO(2)

ρ_{out} , ρ_{in} can be built by block-wise concatenation of smaller, irreducible representations of SO(2)

Irreducible Representations of SO(2):

$$\rho_0(g) = 1, \quad \rho_n(g) = \begin{pmatrix} \cos ng & -\sin ng \\ \sin ng & \cos ng \end{pmatrix}$$

Group Representations of SO(2)

Forming a representation from two irreducible representations

$$\rho = \rho_0 \bigoplus \rho_1$$

$$\rho_0(g) = 1, \quad \rho_n(g) = \begin{pmatrix} \cos ng & -\sin ng \\ \sin ng & \cos ng \end{pmatrix}$$

$$\rho(g) = \begin{pmatrix} 1 & & \\ & \cos g & -\sin g \\ & \sin g & \cos g \end{pmatrix}$$

What Do The Guts Look Like?

$$K_{\text{neigh}}(\theta - g) = \rho_{\text{out}}(-g) K_{\text{neigh}}(\theta) \rho_{\text{in}}(g)$$

N x N block diagonal matrix, made of **your** choice of irreps

M x M block diagonal matrix made of **your** choice of irreps

What Do The Guts Look Like?

$$K_{\text{neigh}}(\theta - g) = \rho_{\text{out}}(-g) K_{\text{neigh}}(\theta) \rho_{\text{in}}(g)$$

N x N block diagonal matrix, made of **your** choice of irreps

N X M matrix

M x M block diagonal matrix made of **your** choice of irreps

What Do The Guts Look Like, Toy Problem

Assume 5-dimensional input features, and 4-dimensional output features

$$K_{\text{neigh}}(\theta - g) = \rho_{\text{out}}(-g) K_{\text{neigh}}(\theta) \rho_{\text{in}}(g)$$

4 x 4 block diagonal matrix, made of **your** choice of irreps

5 x 5 block diagonal matrix made of **your** choice of irreps

What Do The Guts Look Like, Toy Problem

Assume 5-dimensional input features, and 4-dimensional output features

$$K_{\text{neigh}}(\theta - g) = \rho_{\text{out}}(-g) K_{\text{neigh}}(\theta) \rho_{\text{in}}(g)$$

4 x 4 block diagonal matrix, made of **your** choice of irreps

5 x 5 block diagonal matrix made of **your** choice of irreps

$$\rho_{out} = \begin{pmatrix} \rho_1 & 0 \\ 0 & \rho_3 \end{pmatrix} = \begin{pmatrix} cos(-g) & -sin(-g) & 0 & 0 \\ sin(-g) & cos(-g) & 0 & 0 \\ 0 & 0 & cos(-3g) & -sin(-3g) \\ 0 & 0 & sin(-3g) & cos(-3g) \end{pmatrix}$$

What Do The Guts Look Like, Toy Problem

Assume 5-dimensional input features, and 4-dimensional output features

$$K_{\text{neigh}}(\theta - g) = \rho_{\text{out}}(-g) K_{\text{neigh}}(\theta) \rho_{\text{in}}(g)$$

4 x 4 block diagonal matrix, made of **your** choice of irreps

5 x 5 block diagonal matrix made of **your** choice of irreps

$$\rho_{in} = \begin{pmatrix} \rho_0 & 0 & 0 \\ 0 & \rho_1 & 0 \\ 0 & 0 & \rho_1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & cos(g) & -sin(g) & 0 & 0 \\ 0 & sin(g) & cos(g) & 0 & 0 \\ 0 & 0 & 0 & cos(g) & -sin(g) \\ 0 & 0 & 0 & sin(g) & cos(g) \end{pmatrix}$$

What Do The Guts Look Like, Toy Problem

$$K_{\text{neigh}}(\theta - g) = \rho_{\text{out}}(-g)K_{\text{neigh}}(\theta)\rho_{\text{in}}(g)$$

4 x 4 block diagonal matrix, made of **your** choice of irreps

4 X 5 matrix

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What Do The Guts Look Like, Toy Problem

$$K_{\text{neigh}}(\theta - g) = \rho_{\text{out}}(-g) K_{\text{neigh}}(\theta) \rho_{\text{in}}(g)$$

$$\rho_{out} = \begin{pmatrix} \rho_1 & 0 \\ 0 & \rho_3 \end{pmatrix}$$

$$K_{\text{neigh}}(\theta) = \begin{pmatrix} K_{10}(\theta) & K_{11}(\theta) & K_{11}(\theta) \\ K_{30}(\theta) & K_{31}(\theta) & K_{31}(\theta) \end{pmatrix}$$

$$\rho_{in} = \begin{pmatrix} \rho_0 & 0 & 0 \\ 0 & \rho_1 & 0 \\ 0 & 0 & \rho_1 \end{pmatrix}$$

What Do The Guts Look Like, Toy Problem

$$K_{\text{neigh}}(\theta - g) = \rho_{\text{out}}(-g) K_{\text{neigh}}(\theta) \rho_{\text{in}}(g)$$

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$$\rho_{in} = \begin{pmatrix} \rho_0 & 0 & 0 \\ 0 & \rho_1 & 0 \\ 0 & 0 & \rho_1 \end{pmatrix}$$

Solve by looking at each component of $K_{\text{neigh}}(\theta)$ individually!

What Do The Guts Look Like, Toy Problem

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$$\rho_{in} = \begin{pmatrix} \rho_0 & 0 & 0 \\ 0 & \rho_1 & 0 \\ 0 & 0 & \rho_1 \end{pmatrix}$$

Solve by looking at each component of $K_{\text{neigh}}(\theta)$ individually!

Solve in Chunks

$$K_{\text{neigh}}(\theta) = \begin{pmatrix} K_{10}(\theta) & K_{11}(\theta) & K_{11}(\theta) \\ K_{30}(\theta) & K_{31}(\theta) & K_{31}(\theta) \end{pmatrix}$$

$$K_{10}(\theta - g) = \rho_1(-g) K_{10}(\theta) \rho_0(g)$$

$$K_{11}(\theta - g) = \rho_1(-g) K_{11}(\theta) \rho_1(g)$$

$$K_{11}(\theta - g) = \rho_1(-g) K_{11}(\theta) \rho_1(g)$$

$$K_{30}(\theta - g) = \rho_3(-g) K_{30}(\theta) \rho_0(g)$$

$$K_{31}(\theta - g) = \rho_3(-g) K_{31}(\theta) \rho_1(g)$$

$$K_{31}(\theta - g) = \rho_3(-g) K_{31}(\theta) \rho_1(g)$$

Solve in Chunks

$$K_{\text{neigh}}(\theta) = \begin{pmatrix} K_{10}(\theta) & K_{11}(\theta) & K_{11}(\theta) \\ K_{30}(\theta) & K_{31}(\theta) & K_{31}(\theta) \end{pmatrix}$$

$$K_{10}(\theta - g) = \rho_1(-g) K_{10}(\theta) \rho_0(g)$$

$$K_{11}(\theta - g) = \rho_1(-g) K_{11}(\theta) \rho_1(g)$$

$$K_{11}(\theta - g) = \rho_1(-g) K_{11}(\theta) \rho_1(g)$$

$$K_{30}(\theta - g) = \rho_3(-g) K_{30}(\theta) \rho_0(g)$$

$$K_{31}(\theta - g) = \rho_3(-g) K_{31}(\theta) \rho_1(g)$$

$$K_{31}(\theta - g) = \rho_3(-g) K_{31}(\theta) \rho_1(g)$$

Solve in Chunks

$$K_{\text{neigh}}(\theta) = \begin{pmatrix} K_{10}(\theta) & K_{11}(\theta) & K_{11}(\theta) \\ K_{30}(\theta) & K_{31}(\theta) & K_{31}(\theta) \end{pmatrix}$$

$$K_{10}(\theta - g) = \rho_1(-g)K_{10}(\theta)\rho_0(g)$$

Looking for a 2x1 solution to $K_{10}(\theta)$

Solve in Chunks

$$K_{\text{neigh}}(\theta) = \begin{pmatrix} K_{10}(\theta) & K_{11}(\theta) & K_{11}(\theta) \\ K_{30}(\theta) & K_{31}(\theta) & K_{31}(\theta) \end{pmatrix}$$

$$K_{10}(\theta - g) = \rho_1(-g)K_{10}(\theta)\rho_0(g)$$

Solution has basis:

With m=1

$$\begin{pmatrix} \cos m\theta \\ \sin m\theta \end{pmatrix}, \begin{pmatrix} \sin m\theta \\ -\cos m\theta \end{pmatrix}$$

Solve in Chunks

$$K_{\text{neigh}}(\theta) = \begin{pmatrix} K_{10}(\theta) & K_{11}(\theta) & K_{11}(\theta) \\ K_{30}(\theta) & K_{31}(\theta) & K_{31}(\theta) \end{pmatrix}$$

$$K_{10}(\theta - g) = \rho_1(-g)K_{10}(\theta)\rho_0(g)$$

Solution has form

$$K_{10}(\theta) = w_1 \begin{pmatrix} cos(\theta) \\ sin(\theta) \end{pmatrix} + w_2 \begin{pmatrix} sin(\theta) \\ -cos(\theta) \end{pmatrix}$$

Solve in Chunks

$$K_{\text{neigh}}(\theta) = \begin{pmatrix} K_{10}(\theta) & K_{11}(\theta) & K_{11}(\theta) \\ K_{30}(\theta) & K_{31}(\theta) & K_{31}(\theta) \end{pmatrix}$$

$$K_{10}(\theta - g) = \rho_1(-g)K_{10}(\theta)\rho_0(g)$$

$$K_{11}(\theta - g) = \rho_1(-g)K_{11}(\theta)\rho_1(g)$$

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$$K_{30}(\theta - g) = \rho_3(-g)K_{30}(\theta)\rho_0(g)$$

$$K_{31}(\theta - g) = \rho_3(-g)K_{31}(\theta)\rho_1(g)$$

$$K_{31}(\theta - g) = \rho_3(-g)K_{31}(\theta)\rho_1(g)$$

Solve in Chunks

$$K_{\text{neigh}}(\theta) = \begin{pmatrix} K_{10}(\theta) & K_{11}(\theta) & K_{11}(\theta) \\ K_{30}(\theta) & K_{31}(\theta) & K_{31}(\theta) \end{pmatrix}$$

$$K_{31}(\theta - g) = \rho_3(-g)K_{31}(\theta)\rho_1(g)$$

$\rho_{\text{in}} \rightarrow \rho_{\text{out}}$	linearly independent solutions for $K_{\text{neigh}}(\theta)$
$\rho_0 \rightarrow \rho_0$	(1)
$\rho_n \rightarrow \rho_0$	$(\cos n\theta \;\; \sin n\theta)\,,\,(\sin n\theta \;\; -\cos n\theta)$
$\rho_0 \rightarrow \rho_m$	$\begin{pmatrix}\cos m\theta \\ \sin m\theta\end{pmatrix},\begin{pmatrix}\sin m\theta \\ -\cos m\theta\end{pmatrix}$
$\rho_n \rightarrow \rho_m$	$\begin{pmatrix}c_- & -s_- \\ s_- & c_-\end{pmatrix},\begin{pmatrix}s_- & c_- \\ -c_- & s_-\end{pmatrix},\begin{pmatrix}c_+ & s_+ \\ s_+ & -c_+\end{pmatrix},\begin{pmatrix}-s_+ & c_+ \\ c_+ & s_+\end{pmatrix}$

Solve in Chunks

$$K_{\text{neigh}}(\theta) = \begin{pmatrix} K_{10}(\theta) & K_{11}(\theta) & K_{11}(\theta) \\ K_{30}(\theta) & K_{31}(\theta) & K_{31}(\theta) \end{pmatrix}$$

$$K_{31}(\theta - g) = \rho_3(-g) K_{31}(\theta) \rho_1(g)$$

Basis formed by the following matrices

$\rho_{\text{in}} \to \rho_{\text{out}}$	linearly independent solutions for $K_{\text{neigh}}(\theta)$
$\rho_0 \to \rho_0$	(1)
$\rho_n \to \rho_0$	$(\cos n\theta \;\; \sin n\theta)$, $(\sin n\theta \;\; -\cos n\theta)$
$\rho_0 \to \rho_m$	$\begin{pmatrix} \cos m\theta \\ \sin m\theta \end{pmatrix}$, $\begin{pmatrix} \sin m\theta \\ -\cos m\theta \end{pmatrix}$
$\rho_n \to \rho_m$	$\begin{pmatrix} c_- & -s_- \\ s_- & c_- \end{pmatrix}$, $\begin{pmatrix} s_- & c_- \\ -c_- & s_- \end{pmatrix}$, $\begin{pmatrix} c_+ & s_+ \\ s_+ & -c_+ \end{pmatrix}$, $\begin{pmatrix} -s_+ & c_+ \\ c_+ & s_+ \end{pmatrix}$

Solve in Chunks

$$K_{\text{neigh}}(\theta) = \begin{pmatrix} K_{10}(\theta) & K_{11}(\theta) & K_{11}(\theta) \\ K_{30}(\theta) & K_{31}(\theta) & K_{31}(\theta) \end{pmatrix}$$

$$K_{31}(\theta - g) = \rho_3(-g) K_{31}(\theta) \rho_1(g)$$

Solution has form

$$K_{31}(\theta) = w_1 \begin{pmatrix} cos((3-1)\theta) & -sin((3-1)\theta) \\ sin((3-1)\theta) & cos((3-1)\theta) \end{pmatrix}$$

$\rho_{\text{in}} \rightarrow \rho_{\text{out}}$	linearly independent solutions for $K_{\text{neigh}}(\theta)$
$\rho_0 \rightarrow \rho_0$	(1)
$\rho_n \rightarrow \rho_0$	$(\cos n\theta \;\; \sin n\theta)$, $(\sin n\theta \;\; -\cos n\theta)$
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Solve in Chunks

$$K_{\text{neigh}}(\theta) = \begin{pmatrix} K_{10}(\theta) & K_{11}(\theta) & K_{11}(\theta) \\ K_{30}(\theta) & K_{31}(\theta) & K_{31}(\theta) \end{pmatrix}$$

$$K_{31}(\theta - g) = \rho_3(-g)K_{31}(\theta)\rho_1(g)$$

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Solution has form

$$K_{31}(\theta) = w_1 \begin{pmatrix} cos((3-1)\theta) & -sin((3-1)\theta) \\ sin((3-1)\theta) & cos((3-1)\theta) \end{pmatrix}$$

$$+w_2 \begin{pmatrix} sin((3-1)\theta) & cos((3-1)\theta) \\ -cos((3-1)\theta) & sin((3-1)\theta) \end{pmatrix}$$

Solve in Chunks

$$K_{\text{neigh}}(\theta) = \begin{pmatrix} K_{10}(\theta) & K_{11}(\theta) & K_{11}(\theta) \\ K_{30}(\theta) & K_{31}(\theta) & K_{31}(\theta) \end{pmatrix}$$

$$K_{31}(\theta - g) = \rho_3(-g) K_{31}(\theta) \rho_1(g)$$

$\rho_{\text{in}} \to \rho_{\text{out}}$	linearly independent solutions for $K_{\text{neigh}}(\theta)$
$\rho_0 \to \rho_0$	(1)
$\rho_n \to \rho_0$	$(\cos n\theta \ \ \sin n\theta)$, $(\sin n\theta \ \ -\cos n\theta)$
$\rho_0 \to \rho_m$	$\begin{pmatrix} \cos m\theta \\ \sin m\theta \end{pmatrix}$, $\begin{pmatrix} \sin m\theta \\ -\cos m\theta \end{pmatrix}$
$\rho_n \to \rho_m$	$\begin{pmatrix} c_- & -s_- \\ s_- & c_- \end{pmatrix}$, $\begin{pmatrix} s_- & c_- \\ -c_- & s_- \end{pmatrix}$, $\begin{pmatrix} c_+ & s_+ \\ s_+ & -c_+ \end{pmatrix}$, $\begin{pmatrix} -s_+ & c_+ \\ c_+ & s_+ \end{pmatrix}$

Solution has form

$$K_{31}(\theta) = w_1 \begin{pmatrix} cos((3-1)\theta) & -sin((3-1)\theta) \\ sin((3-1)\theta) & cos((3-1)\theta) \end{pmatrix} + w_2 \begin{pmatrix} sin((3-1)\theta) & cos((3-1)\theta) \\ -cos((3-1)\theta) & sin((3-1)\theta) \end{pmatrix} + w_3 \begin{pmatrix} cos((3+1)\theta) & sin((3+1)\theta) \\ sin((3+1)\theta) & -cos((3+1)\theta) \end{pmatrix}$$

Solve in Chunks

$$K_{\text{neigh}}(\theta) = \begin{pmatrix} K_{10}(\theta) & K_{11}(\theta) & K_{11}(\theta) \\ K_{30}(\theta) & K_{31}(\theta) & K_{31}(\theta) \end{pmatrix}$$

$$K_{31}(\theta - g) = \rho_3(-g) K_{31}(\theta) \rho_1(g)$$

$\rho_{\text{in}} \to \rho_{\text{out}}$	linearly independent solutions for $K_{\text{neigh}}(\theta)$
$\rho_0 \to \rho_0$	(1)
$\rho_n \to \rho_0$	$(\cos n\theta \;\; \sin n\theta)\,,\,(\sin n\theta \;\; -\cos n\theta)$
$\rho_0 \to \rho_m$	$\begin{pmatrix} \cos m\theta \\ \sin m\theta \end{pmatrix}, \begin{pmatrix} \sin m\theta \\ -\cos m\theta \end{pmatrix}$
$\rho_n \to \rho_m$	$\begin{pmatrix} c_- & -s_- \\ s_- & c_- \end{pmatrix}, \begin{pmatrix} s_- & c_- \\ -c_- & s_- \end{pmatrix}, \begin{pmatrix} c_+ & s_+ \\ s_+ & -c_+ \end{pmatrix}, \begin{pmatrix} -s_+ & c_+ \\ c_+ & s_+ \end{pmatrix}$

Solution has form

$$K_{31}(\theta) = w_1 \begin{pmatrix} cos((3-1)\theta) & -sin((3-1)\theta) \\ sin((3-1)\theta) & cos((3-1)\theta) \end{pmatrix} + w_2 \begin{pmatrix} sin((3-1)\theta) & cos((3-1)\theta) \\ -cos((3-1)\theta) & sin((3-1)\theta) \end{pmatrix} + w_3 \begin{pmatrix} cos((3+1)\theta) & sin((3+1)\theta) \\ sin((3+1)\theta) & -cos((3+1)\theta) \end{pmatrix} + w_4 \begin{pmatrix} -sin((3+1)\theta) & cos((3+1)\theta) \\ cos((3+1)\theta) & sin((3+1)\theta) \end{pmatrix}$$

Solve in Chunks

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Solution has form

$$K_{31}(\theta) = w_1 \begin{pmatrix} cos((3-1)\theta) & -sin((3-1)\theta) \\ sin((3-1)\theta) & cos((3-1)\theta) \end{pmatrix} + w_2 \begin{pmatrix} sin((3-1)\theta) & cos((3-1)\theta) \\ -cos((3-1)\theta) & sin((3-1)\theta) \end{pmatrix} + w_3 \begin{pmatrix} cos((3+1)\theta) & sin((3+1)\theta) \\ sin((3+1)\theta) & -cos((3+1)\theta) \end{pmatrix} + w_4 \begin{pmatrix} -sin((3+1)\theta) & cos((3+1)\theta) \\ cos((3+1)\theta) & sin((3+1)\theta) \end{pmatrix}$$

Forward Pass

Aggregate features from each neighbor, weighed by **the kernel basis function** and **the learned weight** variable.

$$f'_p \leftarrow \sum_{i,q\in\mathcal{N}_p} w^i_{\mathrm{neigh}} K^i_{\mathrm{neigh}}(\theta_{pq})\rho_{\mathrm{in}}(g_{q\to p})f_q$$

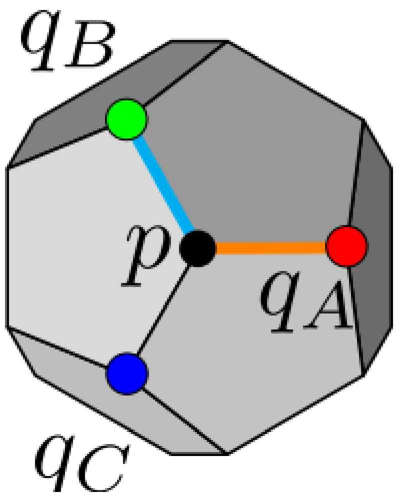
Extra Rotation?

$$f'_p \leftarrow \sum_{i,q\in\mathcal{N}_p} w^i_{\text{neigh}} K^i_{\text{neigh}}(\theta_{pq}) \rho_{\text{in}}(g_{q\to p}) f_q$$

Extra Rotation?

$$f'_p \leftarrow \sum_i w^i_{\text{self}} K^i_{\text{self}} f_p + \sum_{i,q \in \mathcal{N}_p} w^i_{\text{neigh}} K^i_{\text{neigh}}(\theta_{pq}) \rho_{\text{in}}(g_{q \to p}) f_q$$

Features on different vertices live in **different** frames

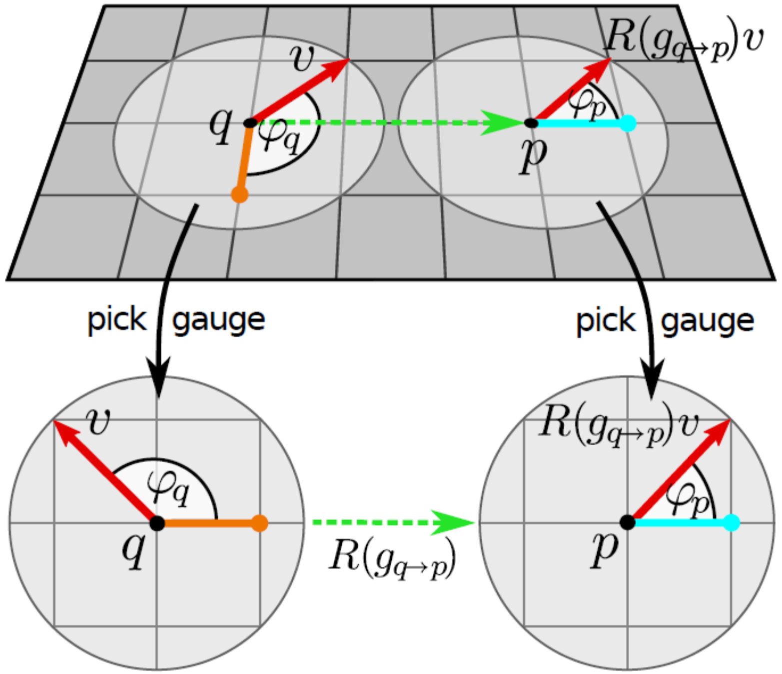


Extra Rotation?

$$f'_p \leftarrow \sum_i w^i_{\text{self}} K^i_{\text{self}} f_p + \sum_{i,q\in\mathcal{N}_p} w^i_{\text{neigh}} K^i_{\text{neigh}}(\theta_{pq}) \rho_{\text{in}}(g_{q\to p}) f_q$$

Features on different vertices live in **different** frames

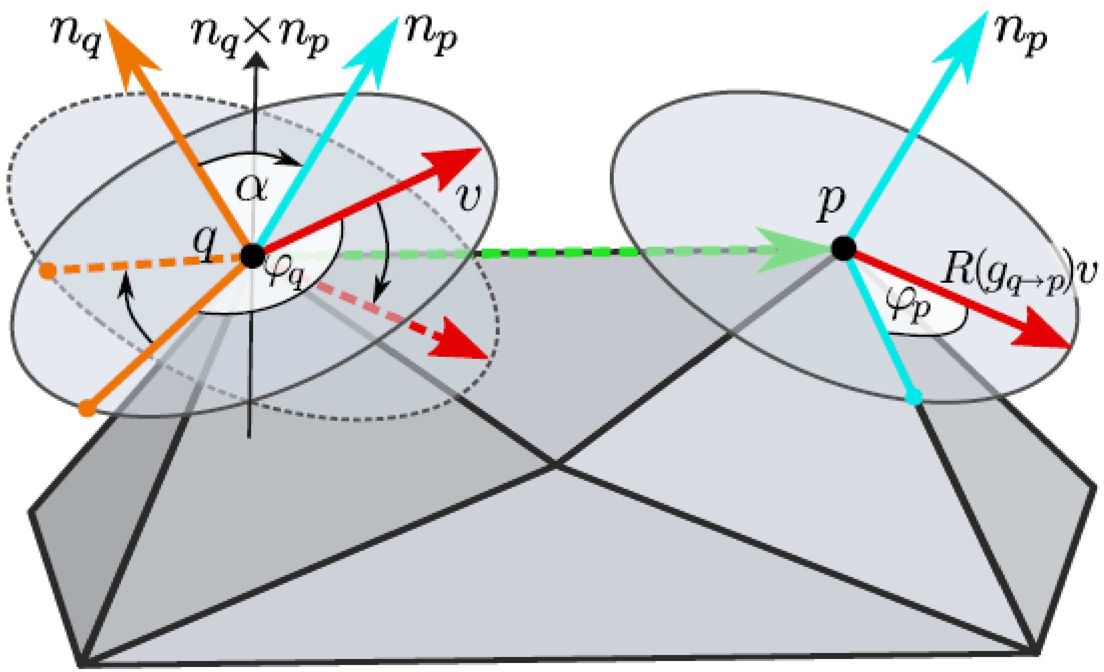
We need to account for this by aligning their frames.



Aligning Frames : Parallel Transport

If mesh is **not** flat, additionally need to **align tangent spaces**, then translate:

- Get axis of rotation by cross product of normals
- Get angle of rotation by dot product of normals
- Form SO(3) rotation matrix.
- Can project all of the steps above into a single 2D gauge transformation.

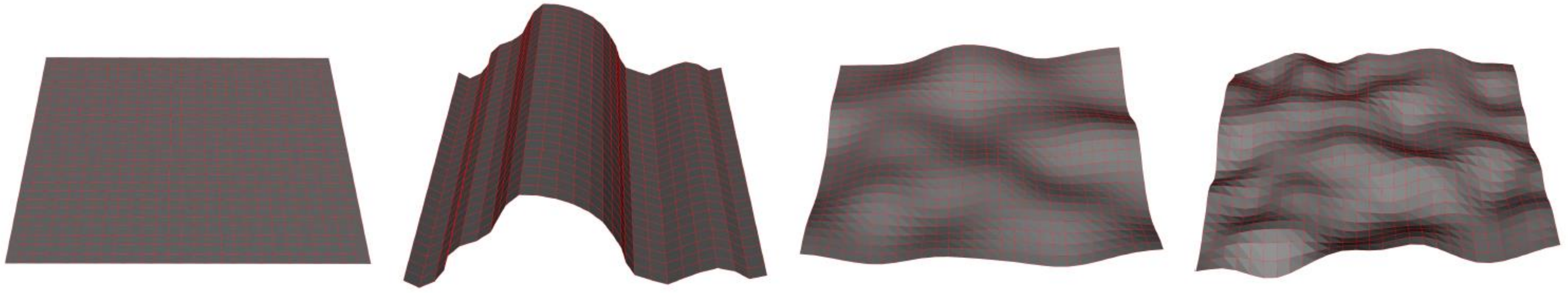


Experimental Results: Embedded MNIST

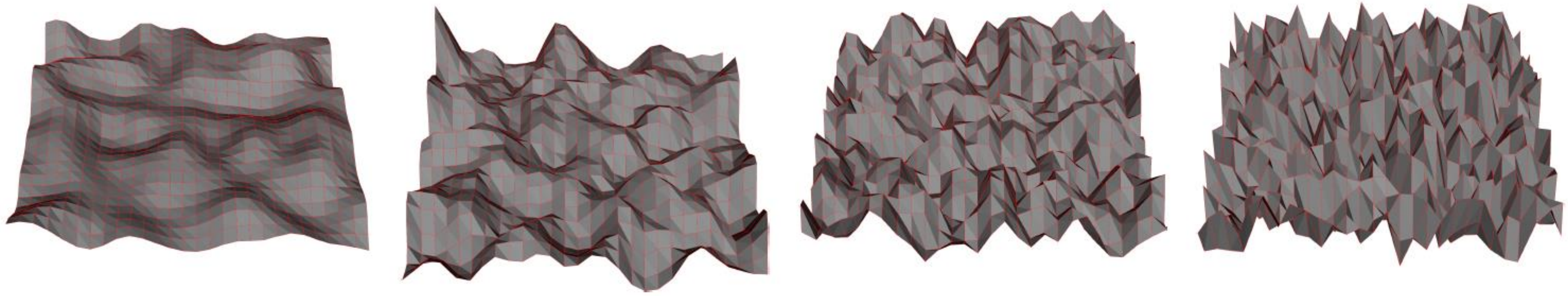
Made a rectangle mesh from MNIST images.

Added random noise to planar mesh. Trained different networks on different mesh roughness.

Experimental Results: Embedded MNIST

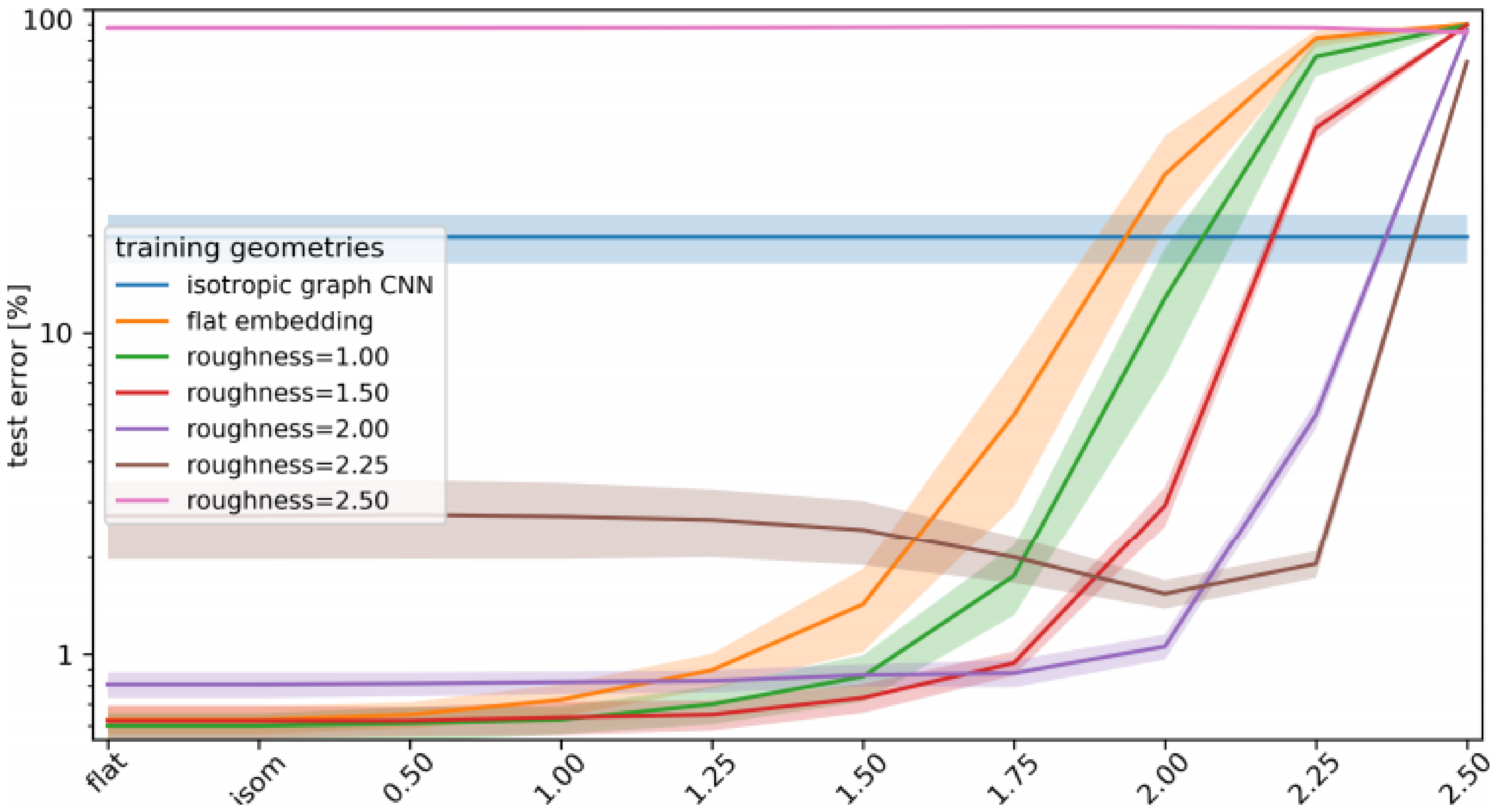


(a) Flat embedding (b) Isometric embedding (c) Curved, roughness = 0.5 (d) Curved, roughness = 1



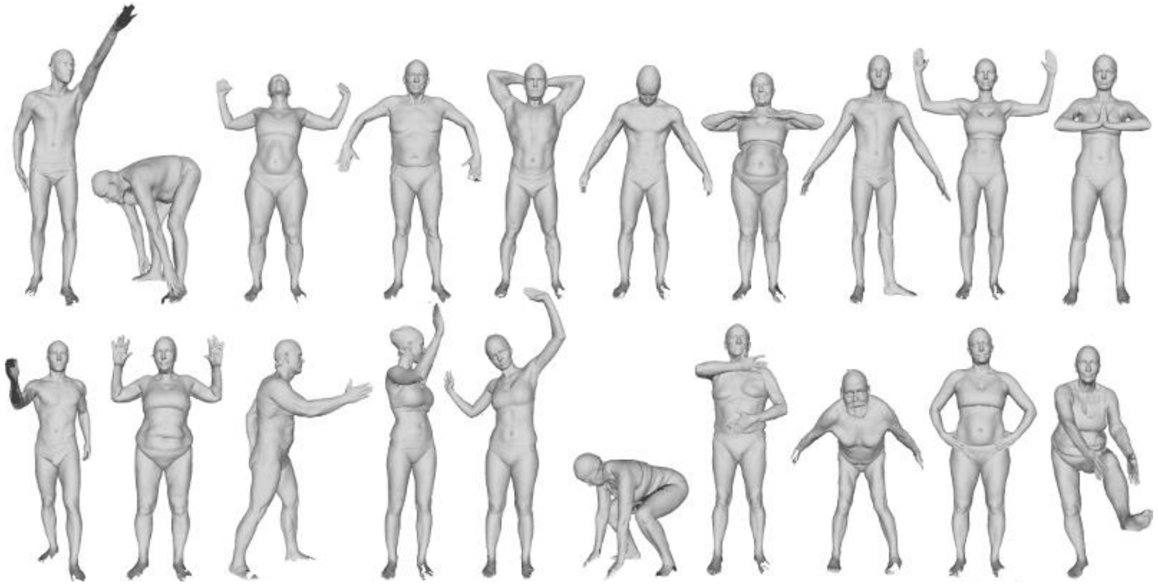
(e) Curved, roughness = 1.5 (f) Curved, roughness = 2 (g) Curved, roughness = 2.25 (h) Curved, roughness = 2.5

Experimental Results: Embedded MNIST



Experimental Results: Shape Correspondance

Given vertex in one human body-mesh, identify corresponding vertex in deformed human body mesh (different pose).



Experimental Results: Shape Correspondance

Model	Features	Accuracy (%)
ACNN (Boscaini et al., 2016)	SHOT	62.4
Geodesic CNN (Masci et al., 2015)	SHOT	65.4
MoNet (Monti et al., 2016)	SHOT	73.8
FeaStNet (Verma et al., 2018)	XYZ	98.7
ZerNet (Sun et al., 2018)	XYZ	96.9
SpiralNet++ (Gong et al., 2019)	XYZ	99.8
Graph CNN	XYZ	1.40±0.5
Graph CNN	SHOT	23.80±8
Non-equiv. CNN (SHOT frames)	XYZ	73.00±4.0
Non-equiv. CNN (SHOT frames)	SHOT	75.11±2.4
GEM-CNN	XYZ	99.73±0.04
GEM-CNN (broken symmetry)	XYZ	**99.89**±0.02

Discussion of results

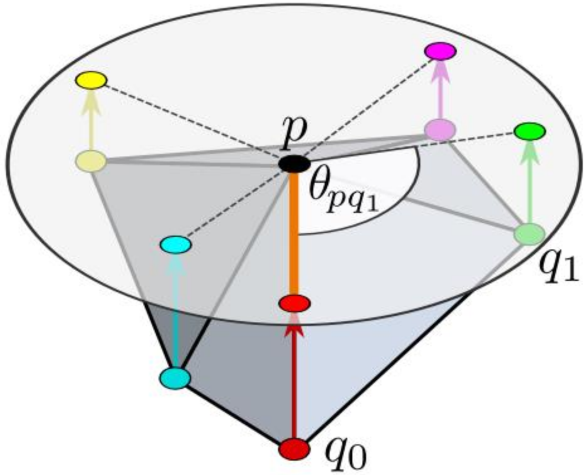
Shows anisotropic graph convolutions are much more expressive than isotropic graph convolutions, on MNIST dataset.

Achieves state of the art performance on shape correspondence with less preprocessing required than other methods, on FAUST dataset.

Critique / Limitations / Open Issues

Still doesn't distinguish between neighborhoods of different curvatures, even if they have the same angular configuration.

Need more general experiments for more convincing argument.



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What Questions Do You Have?